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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Nonlinear Dynamics in Economics and Finance

Jose A. Scheinkman

Phil. Trans. R. Soc. Lond. A 1994 **346**, 235-250
doi: 10.1098/rsta.1994.0020

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By José A. Scheinkman

Department of Economics, University of Chicago, 1126 E. 59th Street, Chicago, Illinois 60637, U.S.A.

The apparent success of 'chaos' in the physical sciences has had inevitable repercussions on economics. In this paper I describe theoretical models that show that complicated deterministic dynamics may arise even in the most standard economic environment, and some of the attempts to evaluate empirically the importance of these nonlinearities in economics and finance. There seems to be no evidence that 'deterministic chaos' can adequately describe economic data, but some evidence of a role for nonlinearities.

1. Introduction

The apparent success of 'chaos' in the physical sciences, has had inevitable repercussions on economics, and some recent theoretical work has focused on the role of nonlinearities in economic dynamics. This is, in a sense, a revival of earlier efforts in the study of economic fluctuations (Kaldor 1940; Hicks 1950; Goodwin 1951) that regarded the market mechanism as dynamically unstable and tried to model economic fluctuations as the output of nonlinear deterministic dynamical systems. However, at least since the early 1960s, the profession had largely switched to a focus on linear (really log–linear) models where exogenous stochastic shocks (e.g. unforecastable changes in technology, in monetary or in fiscal policy) where transformed, through the economy's propagation mechanism, into low-order linear stochastic difference equations, that in turn generated cyclic processes that mimicked actual business cycles.

There seem to have been at least two reasons that led to the dominance of the linear stochastic difference equations approach. The first one was the fact that the nonlinear systems seemed incapable of reproducing some of the most obvious aspects of economic time series. At best, such models were able to produce periodic motion and the examination of the spectra of economic time series showed the absence of the spikes that characterize periodic motion. This objection is now widely known to be invalid in view of results (Sakai & Tokumaru 1980) that show that deterministic systems can generate spectra that would exactly reproduce those of random systems. The second reason was the relative empirical success of the models based on linear stochastic difference equations as well as the lack of evidence of any gains with the introduction of nonlinearities. Examination of asset prices, that economists have long understood to result from many of the same forces that govern output, also seemed to favour stochastic linear models. For the most part a random walk seemed to adequately describe stock returns over periods at least as long as a week (Fama 1970). The development of new algorithms that in principle could detect whether time series that look 'random' were actually the product of low-dimensional

Phil. Trans. R. Soc. Lond. A (1994) **346**, 235–250

Printed in Great Britain

deterministic dynamics, has led economists to re-examine the adequacy of the stochastic linear models.

The interest on nonlinear dynamics goes beyond the theoretical question concerning the intrinsic stability of market economies. Predicting asset returns on the basis of past patterns has been for a long time an active industry, and 'chaotic dynamics' seemed to give it new respectability. Economists have long been sceptical of the possibility of survival for long periods of stable laws that would allow 'technical traders' to exploit profit opportunities. This is a result of viewing market prices as resulting from the actions of economic agents, many of whom are constantly trying to find profit opportunities. In fact, as discussed below, asset prices may, in principle, display chaotic behaviour while, at the same time, no possibility for 'arbitrage' is present.

In this paper I will first describe some of the theoretical models that show that complicated deterministic dynamics may arise even in the most standard economic environment. Though asset returns data lead one naturally to consider economies were a certain amount of 'uncertainty' is present, in these stochastic economies there may still be an important role for nonlinearities, since the conditional distribution of future state variables is, in general, a nonlinear function of the current values of the state variables. The deterministic models can thus still be used to try to understand the economic forces that may be responsible for part of the apparent uncertainty.

Another entirely different matter is whether these potential nonlinearities are needed to explain actual economic data. Although the most successful efforts to confront dynamic equilibrium models with data involve parametrizations where, in the absence of shocks, fluctuations will be absent, there is much left to be explained in the cyclical behaviour of modern economies and in asset prices (Murphy *et al.* 1989). In an attempt to assess the importance of nonlinearities, some researchers have examined economic data using measures developed in the mathematics and physics literature such as the correlation dimension and Liapunov exponents, or using statistical techniques based on these measures. In what follows I will discuss some applications to economic time series. There is no evidence that 'deterministic chaos' can adequately describe economic data, but some evidence of a role for nonlinearities. However, economic forces limit the form of the nonlinearities and, in particular, there has been no successful demonstration that one may produce profitable trading strategies making use of these nonlinearities.

2. Equilibrium models

In modelling economic dynamics we must confront the fact that the agents whose behaviour are generating the dynamics are themselves observing and trying to forecast the actual dynamics. Perhaps this is simpler to understand in the context of stock prices. Suppose that a vector of stock prices follows a dynamics $p_{t+1} = f(p_t, p_{t-1}, \ldots, p_{t-j})$ and that given a particular history of prices $(\bar{p}_t, \bar{p}_{t-1}, \ldots, \bar{p}_{t-j})$ the *rate of return* as of time t, $\bar{p}^i_{t+1}/\bar{p}^i_t$ of an asset i is superior to that of other assets (for simplicity we assume that no dividends are being paid to holders of the asset between t and $t+1$). Speculators that become aware of the dynamics would buy asset i at t and plan to sell it at $t+1$ to profit from this opportunity. This demand by speculators would by itself pressure the price of asset i at t upward, destroying the original dynamics. Though stock speculators may not have available the most sophisticated tools for reconstructing the dynamics from market observations, the continuous observation of market prices must lead them to discover at least some of the profit opportunities and their attempts to profit from the uncovered patterns should alter the dynamics. Of course this does not rule out the possibility that profitable trading patterns may survive for short periods.

Though there is a general agreement among economists that, as in this example, agents' forecasts of future values of the state variables and the effect of these forecasts on the future values of these variables must be modelled explicitly, there is much less consensus on how to accomplish this. At one extreme are the 'rational expectations' or 'perfect foresight' dynamic equilibrium models. Here one tries to derive aggregate behaviour from assumptions on the tastes of individual consumers and technologies available to producers as well as the market structure and by postulating that the economic agents in the model completely understand the structure of the model. In the stock price dynamics discussed above, this approach will require that the law of motion f be such that when agents forecast future prices using f, the demand for assets equals the supply. One should notice that the word *equilibrium* is used here to mean market clearing. The actual price path $\bar{p}_t$ is certainly not constant and may in fact exhibit very complicated dynamics. An important aspect of this 'perfect foresight equilibrium' is the consistency between the law of motion of price that agents perceive and the actual law. This consistency insures that there are no incentives for agents to revise their forecasting rule.

It is, of course highly unlikely that, if prices follow complicated trajectories, agents can learn to forecast perfectly future prices or dividends, even after a long string of observations. The same property that makes chaotic systems look as if they are random – their sensitive dependence to initial conditions – also makes the task for forecasting future values extremely difficult. An outside observer (economist) may very well decide to treat the problem using statistical techniques. The fact that he is mistakenly treating the data as if arising from a random system will not, by itself, invalidate his statistical methods. If the observer allows for a sufficient number of parameters, he can in fact, with enough data, approximate well the true law of motion. If the forecaster uses some simple statistical model, e.g. a linear model, the forecasts will, even in the limit, still display error but in any case the forecasting activity of the economist does not alter the dynamics. In actual economies, however, agents forecast, at each time t, future prices and rates of return and make buying and selling decisions. The aggregate decisions, in turn, affect the rates of return of assets that each agent is trying to learn. In order to define the price dynamics we must postulate the mechanism used by agents to learn about the actual dynamics.

Though some progress has been made concerning equilibrium models with some special learning rules in the presence of exogenous shocks (cf. Bray 1982; Marcet & Sargent 1987; Guesnerie & Woodford 1992), much less is known in the context of deterministic but complicated dynamics. In any case, rational agents facing the problem of forecasting a variable that follows 'chaotic' dynamics, may very well treat it as if the future values of the variables they are trying to predict are at least in part random. In this case we may have to model the economy as if uncertainty is present. From this viewpoint, it may be less crucial to decide whether 'real' uncertainty is present, say through random shocks to the future dividends of the stock in the example discussed above, or whether uncertainty is a result of agents limitations in forecasting. (This has a flavour of the results in the 'sunspot' literature surveyed in Guesnerie & Woodford (1992).) It should be noted, however, that in both cases the realizations of the 'random variables' are likely to affect the future evolution of the state variables. If, for instance, prices of certain assets turn out to be higher than expected at a certain date, agents consumption and savings decisions will be affected. This in turn is likely to affect the future price of the assets. In other words the dynamics of such an economy is given, at best, by a system such as: $p_{t+1} = g(p_t, \mu_t)$ as opposed to a deterministic system in which the observables are subject to random noise.

3. A dynamic Robinson Crusoe economy

The simplest dynamic equilibrium model is one that describes an economy with a fixed technology and homogeneous agents. The homogeneity of agents insures that this economy is essentially a one agent economy much like the *Robinson Crusoe* economy familiar from economic textbooks. This is certainly not realistic, but will allow me, without spending an inordinate amount of space, to raise some of the issues that I want to discuss.

To describe fully such an economy we will need to specify the production possibilities and how consumers evaluate the different consumption streams, i.e. the utility that consumers obtain from consumption sequences. Assume that there is a fixed set of n capital goods and a single consumption good. Consumers are supposed to discount future consumption using a discount factor $0 < \delta < 1$. That is, the representative agent attributes to a sequence $\{c_t\}_{t=0}^{\infty}$ the utility $\Sigma_{t=0}^{\infty} \delta^t c_t$. The production possibilities are the triples (x, y, c) representing the combinations of end of period capital stocks $y \in R_+^n$ and consumption amounts $c \in R_+$ that can be obtained from an initial capital stock vector $x \in R_+^n$ at any period t. I will assume that there exists a compact convex set $B \subset R_+^{2n}$, containing $(0, 0)$, and a C^2, strictly concave function v defined on B, with values on the non-negative reals, such that a triple (x, y, c) is feasible if and only if $(x, y) \in B$ and, $0 \leqslant c_t \leqslant v(x, y)$. Further, v is assumed to be non-decreasing in its first n coordinates and non-increasing in its last n coordinates. The real number $v(x, y)$ measures the maximum utility flow achievable in the period if the initial vector of capital stocks is x and the final vector is y. The monotonicity imposed on v corresponds to the natural hypothesis that in order to achieve a fixed final vector of capital stocks, it is easier to start with a higher capital stock and that it requires consumption sacrifice to accumulate capital. The compactness of B will keep the possible capital stock paths $\{x_t\}_{t=0}^{\infty}$ in a bounded set. (It would be more natural to assume only that $\{y \in R^n \mid \text{there exist } x \in R^n \text{ with } (x, y) \in B\}$ is compact. This, however, would add no generality.) The convexity of B and concavity of v correspond to the absence of increasing returns. This absence of increasing returns is necessary to show that the behaviour of this economy can be mimicked by a market economy as discussed in the next section. With this notation one may write the problem that the representative consumer solves as:

Problem (P):

$$\text{Max} \sum_{0 \leqslant t < \infty} \delta^t v(x_t, x_{t+1}) \quad \text{s.t. } (x_t, x_{t+1}) \in B \quad \text{and} \quad x_0 \geqslant 0 \text{ given.}$$

I will write $V(x)$ for the value of problem (P) when $x_0 = x$. Note that $V(x) < \infty$ and $V(x) > -\infty$, for any x such that there exists a sequence $\{x_t\}_{t=0}^{\infty}$ with $(x_t, x_{t+1}) \in B$ and $x_0 = x$. This will be true, in particular, if $(x, 0) \in B$. The function V satisfies Bellman's equation:

$$V(x) = \max_{(x, y) \in B} \{v(x, y) + \delta V(y)\}. \tag{1}$$

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4. Complicated dynamics

The economy described in the previous section is equivalent to a standard convex dynamic optimization problem. None the less dynamics of the state vector x_t can be remarkably complicated. In fact, the following result by Boldrin & Montrucchio (1986) states that one may arbitrarily choose the dynamics associated with problems such as (P):

Proposition 4.1. *Let X be a compact, convex set in R^n_+, and $f: X \to X$ be a C^2 map. Then there exists convex and compact $B \subset X \times X$, such that for any $x \in X$, there exists $y \in X$ such that $(x, y) \in B$, and smooth strictly concave function $v: B \to R_+$, increasing in the first n coordinates and decreasing in the last n coordinates, and $\delta \in (0,1)$ such that $x_{t+1} = f(x_t)$ defines the unique solution to:*

$$\text{Max} \sum_{0 \leqslant t < \infty} \delta^t v(x_t, x_{t+1}) \quad \text{s.t. } (x_t, x_{t+1}) \in B \quad \textit{and} \quad x_0 \geqslant 0 \textit{ given.}$$

Further, v and δ can be chosen such that if $V(x)$ denotes the value of this problem, then the function $V: X \to R$ is C^2 and strongly concave.

5. Examples

The proof of Proposition 4.1 is constructive and one can exhibit for a given candidate dynamics f, a function v and a discount factor δ, such that $x_{t+1} = f(x_t)$ solves P. Proposition 4.1 shows that complicated dynamics is perfectly consistent with the standard assumptions of competitive models, but it provides little insight into the economic logic that leads to the optimal behaviour being so irregular. To do this one would like to start with certain parametrized classes of examples, where the parameters can, in principle, be matched to data on actual economies, and show that for certain parameter values the resulting *optimal policy function* f is chaotic. This would also help in judging the likelihood that chaotic behaviour arise in actual economies. Since solving problem P explicitly is not possible in many cases, the strategy is to show that, under certain 'natural' assumptions on $\{\delta, v, B\}$ the optimal policy function f is chaotic.

All fully worked out examples in the literature on economic fluctuations arising from purely deterministic models as the one described above (cf. Boldrin & Woodford's (1990) survey and references therein) involve a single capital good, i.e. $n = 1$, even though intuition indicates that multidimensional systems are more likely to give rise to complicated dynamics. When $n = 1$, typically one finds conditions that guarantee the existence of a period three cycle for the optimal policy function f and hence topological chaos. For concave and smooth v, a necessary and sufficient condition for a period three cycle (x_1, x_2, x_3) in the interior of B is that for each $i \in \{1,2,3\}$, if $x_4 = x_1$ and $x_5 = x_2$, the associated Euler equation holds:

$$\frac{\partial v}{\partial y}(x_i, x_{i+1}) + \delta \frac{\partial v}{\partial x}(x_{i+1}, x_{i+2}) = 0. \tag{2}$$

Though equation (2) has been used to show that chaotic dynamics may arise in the context of some parametrized examples, the parameter values that are required are not reasonable. (For instance setting $B = \{(x,y) \in R^2_+ : x \leqslant 1, y \leqslant 1, \gamma y \leqslant x\}$ and, $v(x,y) = (1-y)^\beta (x - \gamma y)^\alpha$ with $0 < \gamma < 1$, $0 < \alpha < 1$, and $0 < \alpha + \beta \leqslant 1$.) In particular the discount rate δ is seldom above 0.3, which indicates a rate of interest above 200 %

per period. (If the utility function v is strongly concave, for δ sufficiently close to 1, all optimal trajectories must be convergent (Scheinkman 1976). Hence complicated dynamics require, other things held equal, a relatively large discount factor. However, other forms of non-convergent behaviour, e.g. periodic orbits have been shown to appear in similar examples at much more reasonable value for δ.) More reasonable examples may be obtainable in the presence of many capital goods, but none have yet been produced. (Several types of 'market imperfections' can also be used to generate complicated dynamics. Another alternative is to deal with 'overlapping generations' models (cf. Boldrin & Woodford 1990).)

6. Price dynamics

As stated, the Boldrin–Montrucchio result deals with the trajectory of capital stocks in a centrally planned economy. One interest here is in the behaviour of market economies and in fact much of the available economic data, specially high frequency data, refer to prices (and sometimes quantities traded) of *assets*. A decentralized version of our artificial economy can, however, be constructed, in a way that is similar to the usual textbook Robinson Crusoe story. In this section I discuss how this decentralized economy can be constructed. More precise statements can be found in Appendix A.

The assumed convexity of the set B and concavity of the utility function v is enough to allow us to characterize a solution to Problem P as resulting from an equilibrium of a dynamic economy. In this economy at each time t there are three sets of markets open. In the first set of markets, capital goods that are used as inputs in the production at t are traded in exchange for the consumption good produced at t. In the second set of markets capital goods that are produced at t are exchanged for the consumption good produced at t. There is also an idealized stock market that I discuss below.

A profit maximizing firm is assumed to own the technology B. At each period t, the firm buys capital goods produced at $t-1$ from consumers, and uses them to produce the consumption good and new capital goods that it sells to consumers. The firm takes prices as given. Notice that the firm faces a purely static problem of profit maximization and we need not make any assumptions concerning how the firm's managers forecast future prices.

The problem for the representative consumer involves, on the other hand, interaction across periods. At each point in time he has to decide how many capital goods of each kind he wants to have next period to sell to firms. This, in turn, requires the consumer to forecast prices that will prevail in the future. Since I have normalized the technology in such a way that capital goods last only one period, to be able to discuss the prices of long lived assets, I will introduce an extra asset market. One share of the firm entitles its owner in period t to receive a dividend that equals the total profit the firm realizes at t. These shares are also traded in a competitive market in exchange for consumption goods.

A representative consumer takes as given the sequence of future prices in all three sets of markets as well as the sequence of future profits that the firm will pay to stockholders. A representative consumer chooses at each $t \geqslant 0$, an amount c_t to consume, an amount x_{t+1} of capital stock to carry into the next period as well as an amount θ_{t+1} of shares to the profits of the firm. In each period t, a consumer's choices must satisfy a budget constraint that states that the consumption and acquisition of

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assets at a period t must be financed by the sale of assets or dividends received. The objective of the consumer is, as before, to maximize $\Sigma_{t=0}^{\infty} \delta^t c_t$.

An equilibrium is a sequence of prices for capital goods inputs and outputs, and the firm's share, today and at each future period, such that if consumers forecast these future prices then their actions will ensure that in all three sets of markets, demand equals supply at every t. This notion of equilibrium makes strong requirements concerning the consumers foresight and missing is any explanation of how consumers would arrive at this forecast, but, at this point, I merely want to show that complicated dynamics may arise even in an economic world in which drastic simplifying assumptions have been made.

As shown in Appendix A, whenever the optimal solution to P is interior, the equilibrium prices of inputs p_t^i satisfy $p_t^i = V'(x_t)$ and hence,

$$p_{t+1}^i = V'(f(V'^{-1}(p_t^i))).$$

Since V is strongly concave, the dynamics of p_t^i is equivalent to that of x_t. Hence the dynamics of the vector of input prices p_t^i can be arbitrarily complicated. The same result holds for either the price of capital goods output at t, or for the dynamics of the share price. Hence the Boldrin–Montrucchio result can be used to establish that in a very simple competitive economy we may obtain, as an equilibrium, complicated paths for the capital stocks, prices of capital goods or stock prices.

These simple dynamic economies do, however, impose other, very strong, restrictions on asset prices. The *rate of return* at time t of an asset is equal to its price plus dividends at $t+1$ divided by its price at t. If, at time t asset l has a higher rate of return than asset l', a consumer that buys asset l' at time t can obtain a higher consumption at $t+1$, without lowering his consumption at any future date, by buying asset l instead of l'. If two assets have a different rate of return, a consumer will not hold any of the asset with the lower rate of return. Hence, in equilibrium, all assets that exist in positive amounts must have the same rate of return. Though asset prices can exhibit complicated dynamics, asset dividends must adjust to equalize returns.

The equality of ex-post returns, which obviously does not hold even as an approximation, can be readily used to reject these purely deterministic models. As discussed above, in these economies consumers are assumed to have *perfect foresight* concerning future prices and dividends. In these perfect foresight economies we are able to show that equilibrium prices can follow very complicated trajectories, but rates of return must be equalized across all assets. As we argued above, it is highly unlikely that, if prices follow complicated trajectories, consumers can forecast perfectly future prices or dividends, and in fact these economies may look as if exogenous stochastic shocks are present. Further, the particular realizations of the 'random variables' are likely to affect the future evolution of the state variables. If, for instance, prices of the capital goods that are used as inputs in the production process turn out to be higher than expected at a certain date, consumers will probably try to save some of their unexpected income. This in turn will affect the supply of capital goods next period and the distribution of future output. In other words the dynamics of such an economy is given, at best, by a system such as: $x_{t+1} = f(x_t, \mu_t)$, as opposed to a deterministic system in which the observables are subject to random noise. (If the technology set B (or the function v) is subject at each t to an independently and identically distributed shock μ_t, then the solution to problem (P) is given by a stochastic difference equation such as $x_{t+1} = f(x_t, \mu_t)$. If we further assume that enough markets exist then we may decentralize the economy much in the same manner as above. In this case asset prices will also follow stochastic difference equations (cf. Lucas 1978).)

The presence of uncertainty does not, however, eliminate the role for nonlinearities. The same forces that create nonlinearities in the deterministic economies I discussed above, can also create nonlinear dependence when randomness is present. (Benhabib & Nishimura (1989) illustrate this point.) Suppose the state variables follow a system such as $x_{t+1} = f(x_t, \mu_t)$ where μ_t is a random variable, and f is nonlinear. Estimating a linear system from the data may lead to an exaggerated view of the role of uncertainty. Another entirely different matter is whether these potential nonlinearities are needed to explain actual economic data. The next section explains some tools developed to answer this question and discusses some empirical results.

7. Statistical tools

The earlier efforts in applying the ideas of chaotic dynamics to uncover nonlinear dependence in economic data (cf. Scheinkman 1985; Brock 1986) consisted simply of using certain tools developed in the mathematics and physics literature in a rather direct way. The application of these techniques to economics present several problems. First, the time series in economics, with perhaps a few exceptions in finance, tend to be much shorter than it seems necessary to obtain good estimates. (Ramsey & Yuan (1987) contains a discussion of the statistical properties of dimension calculation with small data-sets. Ruelle (1989) makes some important admonitions concerning dimension calculation.) Secondly, as I argued above, it is unlikely that the economic time series of interest are generated by purely deterministic systems. Further, the uncertainty is likely to affect the dynamics itself as opposed to merely affecting the observations. None the less the earlier work suggested that nonlinearities may be present in certain economic time series and inspired the development of asymptotic distribution theory for some statistics related to the correlation dimension (cf. Brock *et al.* 1987). (There are of course many other statistical techniques designed to detect the presence of nonlinearities that have been applied to economic time series (cf. Engle 1982; Hinich 1982; Tsay 1986). However, I focus here exclusively on methods related to nonlinear dynamics.)

Let $y_1, y_2, \ldots,$ be a sequence of vectors in R^p. For each $\gamma > 0$, let

$$C_m = \left(\frac{2}{m(m-1)}\right) \sum_{1 \leqslant i < j \leqslant m} \theta(\gamma - |y_i - y_j|), \tag{3}$$

where $\theta(a) = 0$ if $a < 0$, and $\theta(a) = 1$ if $a \geqslant 0$. Here, $|y_i - y_j| = \max_k |y_i^k - y_j^k|$.

Intuitively $C_m(\gamma)$ denotes the fraction of the first m vectors y_is that are within γ of each other. For each γ, if the limit exists, let

$$C(\gamma) = \lim_{m \to \infty} C_m(\gamma). \tag{4}$$

The quantity $C(\gamma)$ indicates the fraction of all vectors that are within γ of each other. (Recall that the correlation dimension of $\{y_t\}_{t=0}^{\infty}$ is defined as $d = \lim_{\gamma \to 0} \log C(\gamma)/\log \gamma$.) If $x_1, x_2, \ldots$ is a sequence of real numbers, for $N \geqslant 1$ each $z_t^N = (x_t, x_{t+1}, \ldots, x_{t+N-1})$ will be called an N-history. For each m, let

$$C_m^N(\gamma) = \left(\frac{2}{m(m-1)}\right) \sum_{1 \leqslant i < j \leqslant m} \prod_{k=0}^{N-1} [\theta(\gamma - |x_{i+k} - x_{j+k}|)], \tag{5}$$

and, if the limit exists,
$$C^N(\gamma) = \lim_{m\to\infty} C_m^N(\gamma). \tag{6}$$

Clearly, $C_m^N(\gamma)$ is the fraction of the first m, N-histories that are within γ of each other and similarly for $C^N(\gamma)$.

If each x_t is an observation of independently and identically distributed (IID) random variables then one should expect that, for m large,

$$C_m^N(\gamma) = [C_m^1(\gamma)]^N. \tag{7}$$

In fact, it can be shown (see Appendix B for details) that there exists a sequence of positive numbers $V_{N,m}$, that can be computed from the data, such that if each x_t is an observation of IDD random variables, then

$$W_m^N(\gamma) = \surd m[C_m^N(\gamma) - (C_m^1(\gamma))^N]/V_{N,m} \tag{8}$$

is asymptotically distributed as a normal distribution with mean zero and unit variance. This distribution free statistics can be used in testing for the presence of nonlinearities. A key point in establishing (8) is the recognition that $C_m^N(\gamma)$ is a U-statistic in the sense of Hoeffding (1948). (Originally U-statistics were defined for the case where $y_1, y_2, \ldots$, are IID. A symmetric function $h: R^N \to R$ is a kernel for μ if $\mu = Eh(y_1, \ldots, y_N)$. Corresponding to the kernel h there is a U-statistic

$$U(y_1, \ldots, y_m) = \binom{m}{N}^{-1} \Sigma\, h(y_{i_1}, \ldots, y_{i_N})$$

where the summation is over all $\binom{m}{N}$ combinations of N distinct elements $(i_1, \ldots, i_N)$ from $\{1, \ldots, m\}$.) Several results concerning central limit theorem for U-statistics exist in the literature. Modern treatments of the theory of U-statistics can be found in Sefling (1980) and Denker & Keller (1983). The fact that $C_m^N(\gamma)$ is U-statistic can also be used to show asymptotically normal behaviour of other statistics related to the correlation dimension, including estimates of the slope of $\log[C_m^N(\gamma)]$.

Frequently one is interested in finding nonlinear dependence on the residuals of particular models fitted to the data. In many macroeconomic time series, for example, low-order autoregressive models are known to yield a good fit. In the analyses of foreign exchange rates, ARCH models (cf. Engle 1982) were used by Hsieh (1989) to pre-filter the data. In practice one can proceed as in Scheinkman & LeBaron (1989*a*, *b*) to examine the distribution of the estimated residuals. First, the model is estimated and a set of residuals is generated. These residuals are randomly reordered and data-sets are then reconstructed using the estimated model. In each of these data sets one re-estimates the model and measures the $W_m^N(\gamma)$ statistics on the residuals. This 'bootstrap' like procedure is then used to determine the significance of the value of the statistics in the original residuals. Another possibility is to determine the effect on the variance of the estimator caused by the fact that the residuals are estimated. Results of this type can be found in Brock *et al.* (1990).

8. Empirical results

The statistics discussed in the preceding section as well as methods developed in the mathematics and physics literature, have been used to detect evidence of chaos, or at least for the presence of nonlinearities, in economic time series.

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In macroeconomics, time series are simply too short and noisy. Most macroeconomic data have a quarterly or at most monthly frequency. There are a few economic time series that have been produced for a very long period but, in these cases, there is usually strong evidence against stationarity. Statistics such as (8) above, as well as others, have been used to detect evidence for nonlinearity in economic time series including U.S. industrial production and unemployment series.

Financial time series seem immune to many of the problems that plague macroeconomic data. First, prices of assets are observed rather directly and this avoids many of the measurement issues concerning macroeconomic time series. Secondly, data are sampled at high frequencies. Nonetheless the application of tests developed in the mathematics and physics literature have not led to any convincing evidence for deterministic chaos.

Attempts to use statistics based on the correlation dimension, such as (8), to find evidence for the presence of nonlinearities have met with more success. Scheinkman & LeBaron (1989*a*) found strong evidence to reject the hypothesis of IID innovations in weekly and daily returns on the value weighted portfolio from CRSP. (The Center for Research in Securities Prices at the University of Chicago.) Potential source of these results were several calendar anomalies that had been detected earlier in stock returns including monthly by Ariel (1987) and weekly by French (1980). LeBaron (1988) found that accounting for these anomalies did not alter the reported rejection of the hypothesis of IID innovations.

Fama (1965) following a suggestion of Mandelbrot, found some evidence that large absolute price changes in certain stock prices tended to be followed by other large absolute price changes. The ARCH model and its GARCH variant (cf. Engle 1982; Bollerslev 1986) provide parametric versions that capture conditional heteroskedasticity, that is the fact that the variance of the distribution of price changes, conditional on past prices, is not constant. Hsieh (1990) discusses the literature and presents further evidence indicating the presence of conditional heteroskedasticity in weekly and daily stock returns. To examine whether these models could accommodate the observed departures from IID innovations, LeBaron (1991) looked at GARCH residuals of the weekly returns on the CRSP value weighted portfolio. The estimates based on the whole sample (1962–86) showed that the normalized GARCH residuals continued to show significant departures from IID innovations. However, if the sample is divided in two halves, one cannot reject the IID hypothesis on the residuals of the second half. This result raises questions concerning the stationarity of the return series, which is frequently an implicit assumption.

The departure from IID residuals would seem to indicate that a prediction of returns superior to the forecasts implied by the random walk model could be made. LeBaron (1988) tried locally weighted regressions and was unable to beat the random walk prediction in a convincing manner. His results seem to indicate that the departures from the random walk may occur principally through the forecastability of the second or higher moments. Mayfield & Mizrach (1989) also examined the question of predictability in intra-day data. They examined data on the S & P 500 stocks average, sampled once every twenty seconds and concluded that predictions superior to random walk predictions could not be made even five minutes ahead.

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9. Conclusion

In this paper I surveyed the impact of some of the recent developments in the mathematics of nonlinear deterministic time evolutions in economics. From a theoretical viewpoint it has helped economists understand that, at least in principle, some of the apparent randomness in economic time series may be the result of the presence of nonlinearities. Eventually this may prove to be a fruitful approach in explaining the observed instability of economic time series, but this must wait the development of more 'realistic' theoretical models. Examination of asset returns indicates the presence of nonlinear dependence in several financial time series, though persuasive evidence to favour chaos has not been uncovered.

The potential use of dynamical systems methods to predict future asset prices seemed to give new respectability to 'technical trading'. The published literature to date does not support this view though there is, of course, the possibility that methods of dynamical systems may be used to quickly detect temporary patterns that may appear in asset price series. These results do not surprise most economists. There is a lot of resources dedicated to finding strategies that yield extraordinary returns. As such opportunities are detected, the effect on prices of the collective actions of agents trying to make profits tends to destroy the existing patterns. If at some point in time, nonlinearities allow for strategies that generate abnormal profits, market forces should eventually lead to a change in the dynamics of stock prices.

Appendix A

In this appendix I state more formally the results discussed in §5. If one assumes that the technology set B is 'sufficiently productive', the postulated convexity (i.e. the convexity of B and the concavity of the function v) is enough (cf. McKenzie 1976) to guarantee that if the sequence x_t, $t \geqslant 0$ solves (P), there exists a sequence $q_t \in R^n_+$ such that, for each t, and each $(x, y) \in B$

$$\delta^t v(x_t, x_{t+1}) + q_{t+1} \cdot x_{t+1} - q_t \cdot x_t \geqslant \delta^t v(x, y) + q_{t+1} \cdot y - q_t \cdot x. \tag{9}$$

Further, q_t/δ^t is a subgradient of the concave function V at x_t. (With the assumptions we have made, it suffices to add the hypothesis that $f(X)$ has a non-empty interior, and that x_0 is in the interior of X. Here, $f: X \to X$ is the candidate optimal policy function. McKenzie (1976) contains much weaker hypothesis.)

This result can be used to give a 'decentralized' interpretation to our economy. A profit maximizing firm owns the technology. At each period t the firm buys capital goods produced at $t-1$ from consumers, and uses them to produce the consumption good as well as new capital goods that it sells to consumers. I denote all period t prices in terms of period t consumption good. Write p^{i}_t for the price of the capital good that is used as input to the production process at t and p^{o}_t for the price of the capital goods that are produced at t. The firm takes the prices as given, and chooses among the feasible production plans the one that maximizes profits. That is, the firm solves:

$$\operatorname{Max}\{v(x, y) + p^{\mathrm{o}}_t \cdot y - p^{\mathrm{i}}_t \cdot x\} \text{ s.t. } (x, y) \in B. \tag{10}$$

Since v is strictly concave, there is a unique (x, y) that solves (10). Write the solution as $(z^{\mathrm{i}}(p^{\mathrm{i}}_t, p^{\mathrm{o}}_t), z^{\mathrm{o}}(p^{\mathrm{i}}_t, p^{\mathrm{o}}_t))$. Let $\pi(p^{\mathrm{i}}_t, p^{\mathrm{o}}_t)$ denote the maximal value of (10).

One share of the firm entitles its owner in period t to receive a dividend that equals the total profit the firm realizes at t. These shares are traded in a competitive market,

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and let s_t denote its time t price, after the dividend is paid. Again this price is in terms of time t consumption good.

The consumer takes as given the sequence of prices $(p_t^{\mathrm{i}}, p_t^{\mathrm{o}}, s_t)$ for $t \geqslant 0$. He also takes as given the sequence of profits π_t that the firm will pay to stockholders, at each $t \geqslant 0$. The representative consumer chooses at each $t \geqslant 0$, an amount c_t to consume, an amount x_{t+1} of capital stock to carry into the next period as well as an amount θ_{t+1} of shares to the profits of the firm. In each period t, the consumer's choices must satisfy the budget constraint

$$c_t + p_t^{\mathrm{o}} \cdot x_{t+1} + s_t(\theta_{t+1} - \theta_t) \leqslant p_t^{\mathrm{i}} \cdot x_t + \pi_t \theta_t. \tag{11}$$

The right-hand side of (11) defines the income of the consumer at t. It consists of the sale of capital goods to the firm and of dividends that are proportional to his stockholdings. The left-hand side gives us his total expenditures: consumption, acquisition of capital goods for future sales and changes in his stock positions. Let, $s = \{s_t\}_{t=0}^{\infty}$, $\pi = (\pi_t\}_{t=0}^{\infty}$, etc. The objective of the representative consumer is to solve:

Problem $Q(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}})$

$$\operatorname{Max} \sum_{0 \leqslant t < \infty} \delta^t c_t$$

$$\text{s.t. (11) and } x_0 \geqslant 0 \text{ given}, \quad \theta_0 = 1, \quad x_t \geqslant 0 \quad \text{and} \quad \theta_t \geqslant 0. \tag{12}$$

Let $c_t(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}})$, $x_t(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}})$ and $\theta_t(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}})$ for $t \geqslant 0$ denote the solution to Problem $Q(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}})$, if it exists.

An equilibrium is a triple of non-negative sequences, $p^{\mathrm{i}}, p^{\mathrm{o}}, s$, such that, for each $t \geqslant 0$,

$$\theta_t(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}}) = 1, \tag{13}$$

$$x_t(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}}) = z^{\mathrm{i}}(p_t^{\mathrm{i}}, p_t^{\mathrm{o}}), \tag{14}$$

$$x_{t+1}(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}}) = z^{\mathrm{o}}(p_t^{\mathrm{i}}, p_t^{\mathrm{o}}), \tag{15}$$

$$c_t(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}}) = v(z^{i}(p_t^{\mathrm{i}}, p_t^{\mathrm{o}}), z^{\mathrm{o}}(p_t^{\mathrm{i}}, p_t^{\mathrm{o}})). \tag{16}$$

Equation (13) guarantees the clearing of the stock market. Equation (14) states that the quantity of capital demanded by the firms for current production equals the quantity offered by the representative consumer. Equation (15) states that the amount of capital goods produced by the firm equals the amount that the consumer desires to buy. Finally equation (16) guarantees that the consumption good market is in equilibrium. In words, an equilibrium is a sequence of prices for capital goods inputs and outputs, and the stock, today and at each future period, such that if consumers forecast these future prices then their actions will ensure that markets will clear at these prices at every t.

Though an equilibrium involves an infinite number of equations, it is easy to characterize an equilibrium in the Robinson Crusoe economy by making use of the 'support property' (9). In fact, one may prove:

Proposition A 1. *Suppose* x_t *solves* (P) *and* q_t *are the support prices defined by* (9). *Then the sequences* p^{i}, p^{o} *and* s *given by,*

$$p_t^{\mathrm{i}} = q_t / \delta^t, \tag{17}$$

$$p_t^{\mathrm{o}} = q_{t+1} / \delta^t, \tag{18}$$

$$s_t = V(x_t) - (q_t / \delta^t) \cdot x_t - \pi(q_t / \delta^t, q_{t+1} / \delta^t), \tag{19}$$

form an equilibrium. In this equilibrium, for each $t \geqslant 0$, $x_t(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}}) = x_t$, *and* $c_t(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}}) = v(x_t, x_{t+1})$.

Proof. From (9), it is clear that $z^{\mathrm{i}}(q_t/\delta^t, q_{t+1}/\delta^t) = x_t$, and $z^{\mathrm{o}}(q_t/\delta^t, q_{t+1}/\delta^t) = x_{t+1}$. It suffices then to show that $c_t = v(x_t, x_{t+1})$, x_t and $\theta_t = 1$ for $t \geqslant 0$ solve $\mathrm{P}(s, \pi, p^{\mathrm{i}}, p^{\mathrm{o}})$. Note that

$$\delta p^{\mathrm{i}}_{t+1} = p^{\mathrm{o}}_t, \tag{20}$$

$$\delta(\pi_{t+1} + s_{t+1}) = s_t. \tag{21}$$

Since, $p^{\mathrm{o}}_t/\delta \in \partial V(x_{t+1})$, and $V \geqslant 0$, $p^{\mathrm{o}}_t \cdot x_{t+1} \leqslant V(x_{t+1})$ and, $s_t \leqslant V(x_t)$ and V is bounded, it follows that $\lim_{t\to\infty} \delta^t(p^{\mathrm{o}}_t \cdot x_{t+1} + s_t) = 0$. This combined with (20) and (21) guarantees the optimality of $c_t = v(x_t, x_{t+1})$, x_t and $\theta_t = 1$.

This result allows us to relate equilibrium prices to the optimal path $x_{t+1} = f(x_t)$ obtained in Proposition 4.1. If x_t is an interior path, since V is strongly concave, p^{i}_t will satisfy

$$p^{\mathrm{i}}_{t+1} = V'(f(V'^{-1}(p^{\mathrm{i}}_t))) \tag{22}$$

and this dynamics is equivalent to that of x_t. Clearly, p^{o}_t has the same property. Further, from (19), we have that if x_t is interior,

$$s_t = V(x_t) - (q_t/\delta^t) \cdot x_t - \pi(q_t/\delta^t, q_{t+1}/\delta^t) = \delta[V(x_{t+1}) - V'(x_{t+1}) \cdot x_{t+1}].$$

Since V is strongly concave, if $n = 1$, we may again write,

$$s_{t+1} = g(s_t),$$

and the dynamics of s_t is equivalent to that of x_t. (When $n > 1$ one could attempt to show, as in Takens (1983), that 'generically, observations on the histories of s_t are equivalent to observations of x_t. The difficulty here is that the function that determines the 'observable' cannot be chosen independently of the dynamics f.) The combination of Propositions 4.1 and Proposition A 1, can be used to show that even in very simple dynamic economies prices can follow very complicated trajectories. These simple dynamic economies do, however, impose other, very strong, restrictions. The *rate of return* between t and $t+1$ of an asset is equal to its price plus dividends at $t+1$ divided by its price at t. Equation (21) above states that the rate of return of the stock equals $1/\delta$. Similarly (20) states that an agent that buys the capital good j, $j = 1, 2, \ldots, n$, at t, pay $p^{\mathrm{o}j}_t$ and sells it at $t+1$ for $p^{\mathrm{i}j}_{t+1}$ again has a rate of return of $1/\delta$. The fact that the rate of return is constant over time is an artifice of the choice of *numeraire*, the units in which prices are measured. If prices were measured in say, the price at which the firm sells the first capital good in each period, $p^{\mathrm{o}1}_t$, this property would no longer hold, except in very special cases. With a different numeraire, rates of return would fluctuate over time. On the other hand, the fact that the rate of return, at each time t, is the same for all assets that are held in equilibrium is simply a consequence of the consumer's optimization. If two assets have a different rate of return, a consumer will not hold any of the asset with the lower rate of return. Though asset prices can exhibit complicated dynamics, asset dividends must adjust to equalize returns.

Appendix B

This appendix contains further details concerning the asymptotic distribution theory for some statistics related to the correlation dimension. I will follow the exposition in Brock *et al.* (1990).

Let F be the common distribution of the x_ts and,

$$C = \iint \theta(\gamma - |u-v|)\,\mathrm{d}F(u)\,\mathrm{d}F(v), \tag{23}$$

$$K = \int [F(u+\gamma) - F(u-\gamma)]^2\,\mathrm{d}F(u). \tag{24}$$

Notice that $K \geqslant C^2$, and Dechert (1988) showed that unless $F[u+\gamma]-F[u-\gamma]$ is constant then $K > C^2$. I can now state:

Proposition B 1. *If $\{x_t\}_{t=1}^{\infty}$ is* IID *and if $K > C^2$, then for $N \geqslant 2$, as $m \to \infty$*

$$\surd m\,[C_m^N(\gamma) - (C_m^1(\gamma))^N]/V_N \to N(0,1), \tag{25}$$

where

$$V_N^2 = 4[N(N-2)\,C^{2N-2}(K-C^2) + K^N - C^{2N}] + 8 \sum_{j=1}^{N-1} [C^{2j}(K^{N-j} - C^{2N-2j}) - NC^{(2N-2)}(K-C^2)]. \tag{26}$$

Proof. Notice that $C_m^N(\gamma) - [C_m^1(\gamma)]^N = g[C_m^N(\gamma), C_m^1(\gamma)]$, where $g(u,v) = u - v^N$. Since $\surd m\,G(X_m)$ is asymptotically normal if $\surd m\,X_m \to N(0,\Sigma)$ and G has non-zero gradients (this uses what is known in statistics as the delta method) it suffices to show that

$$\surd m\,([C_m^N(\gamma), C_m^1(\gamma)] - [C^N, C]) \to N(0,\Sigma). \tag{27}$$

More precisely from (27) it follows that (cf. Serfling 1980, ch. 3),

$$\surd m\,\{C_m^N(\gamma) - [C_m^1(\gamma)]^N\} \to N(0,\sigma),$$

where $$\sigma = \left[\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}\right]' \Sigma \left[\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}\right] = \{1, NC^{N-1}\}'\,\Sigma\{1, NC^{N-1}\}.$$

In turn, (27) follows if one can show that for any pair (λ_1, λ_2),

$$\surd m\,\{\lambda_1\,C_m^N(\gamma) + \lambda_2\,C_m^1(\gamma) - \lambda_1\,C^N - \lambda_2\,C\} \to N(0, \sigma(\lambda_1,\lambda_2)). \tag{28}$$

In fact in this case, the entries of Σ are given by, $\Sigma_{11} = \sigma(1,0)$, $\Sigma_{22} = \sigma(0,1)$, $\Sigma_{12} = \Sigma_{21} = \frac{1}{2}[\sigma(1,1) - \sigma(1,0) - \sigma(0,1)]$.

To show that (28) holds, note that for any pair λ_1, λ_2 and for any realization of the histories $z_1, z_2, \ldots, z_m$ we can write,

$$\lambda_1\,C_m^N(\gamma) + \lambda_2\,C_m^1(\gamma) = [2/m(m-1)] \sum_{1 \leqslant i < j \leqslant m} h(z_i, z_j), \tag{29}$$

where $$h(z_i, z_j) = \lambda_1 \prod_{k=0}^{N-1} \theta(\gamma - |x_{i+k} - x_{j+k}|) + \lambda_2\,\theta(\gamma - |x_i - x_j|),$$

and $z_i = (x_i, x_{i+1}, \ldots, x_{i+N-1})$. The function h is symmetric, i.e. $h(\xi_1,\xi_2) = h(\xi_2,\xi_1)$ for any pair of vectors $(\xi_1,\xi_2) \in R^{2N}$. Hence $\lambda_1\,C_m^N(\gamma) + \lambda_2\,C_m^1(\gamma)$ is a U-statistic, and furthermore, even though two arbitrary histories z_t and z_τ are not in general independent, they will be so if $|t-\tau| > N$. Hence the theorems of Sen (1963) or Denker & Keller (1983) apply and since $Eh(z_i,z_j) = \lambda_1\,C^N + \lambda_2\,C$ if $|i-j| > N$, $\surd m\,\{\lambda_1\,C_m^N(\gamma) + \lambda_2\,C_m^1(\gamma) - \lambda_1\,C^N - \lambda_2\,C\}$ is asymptotically normal. The formula from Sen (1963) can be used to compute the variance $\sigma(\lambda_1,\lambda_2)$ and to obtain (26).

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In the above proposition the constants C and K involve the actual distribution F. However, let

$$C_m = \frac{1}{m^2} \sum_{s=1}^{m} \sum_{t=1}^{m} \theta(\gamma - |x_s - x_t|), \tag{30}$$

and
$$K_m = \frac{1}{m^3} \sum_{r=1}^{m} \sum_{s=1}^{m} \sum_{t=1}^{m} \theta(\gamma - |x_r - x_s|)\, \theta(\gamma - |x_s - x_t|), \tag{31}$$

and $V^2_{N,m}$ equals the right-hand side of (26) when C_m replaces C and K_m replaces K. We have then:

Corollary B 1. *Under the conditions of Proposition B* 1, *as* $m \to \infty$,

$$\sqrt{m}\,[C^N_m(\gamma) - (C^1_m(\gamma))^N]/V_{N,m} \to N(0,1).$$

Proof. C_m and K_m are V-statistics (cf. Serfling 1980) which converge almost surely to C and K respectively. Thus $V_{N,m}$ converges a.s. to V and by Slutsky's theorem and Proposition B 1 above we have the result.

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